Sparsity and Structured Sparsity in Machine Learning:

Models, Algorithms, and Applications

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Outline

Notation

Many machine learning (ML) problems involve learning a mapping from one space to another. Notation:

- Input set $\mathcal{X}$
- For each $x \in \mathcal{X}$, candidate outputs are $\mathcal{Y}(x) \subseteq \mathcal{Y}$
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x	Only RB	France NNP	and CC	Britain NNP	backed VBD	Fischler NNP	's POS	proposal NN
y = $h_{\mathbf{w}}$(x)	O	I-LOC	O	I-LOC	O	I-PER	O	O

Linear Models

Our predictor will take the form

$$h_{\mathbf{w}}(x) = \arg \max_{y \in \mathcal{Y}(x)} \mathbf{w}^\top \mathbf{f}(x, y)$$

where:

- $\mathbf{f}$ is a vector function that encodes all the relevant things about (x, y); the result of a theory, our knowledge, feature engineering, unsupervised feature learning, etc.
- $\mathbf{w} \in \mathbb{R}^D$ are the weights that parameterize the mapping.

Often, *e.g.*, in *natural language processing* (NLP) D is very large ($> 10^6$).

In some cases (*e.g.*, in NLP), the maximization above is itself challenging.

Learning Linear Models

- We observe a collection of examples $\{(x_n, y_n)\}_{n=1}^{N}$.

$y = \text{mountain}$

$y = \text{coast}$

- **Learn** $\mathbf{w}$ from the data, via regularized empirical risk minimization,

$$\widehat{\mathbf{w}} = \arg\min_{\mathbf{w}} \underbrace{\frac{1}{N}\sum_{n=1}^{N} L(\mathbf{w}; x_n, y_n)}_{\text{empirical risk}} + \underbrace{\Omega(\mathbf{w})}_{\text{regularizer}}$$

Logistic regression, perceptron, conditional random fields, SVM, some supervised generative models, all fit the linear modeling framework.

What is Sparsity? Why Is It Important?

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- Models with just a few features are:
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  - cheaper to implement
- Generalization:
  - the goal of ML is to generalize well to new examples
  - encouraging the use of few features discourages overfitting

(Automatic) Feature Selection

Domain experts are often good at engineering features.

Can we automate the process of selecting which ones to keep?

Three main classes of methods (Guyon and Elisseeff, 2003):

1. filters
2. wrappers
3. embedded methods

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3. embedded methods **(this talk)**

Embedded Methods for Feature Selection

Formulate the learning problem as a trade-off between

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Key advantage: declarative statements of model "desirability" often lead to well-understood, solvable optimization problems.

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Loss functions (I)

- Regression ($y \in \mathbb{R}$) typically uses the **squared error** loss:

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- The most/best studied loss function (statistics, machine learning, signal processing); dates back to Legendre and Gauss (early 1800s).

Loss functions (II)

- Classification and structured prediction using **log-linear models** (logistic regression, max ent, conditional random fields):

$$
\begin{aligned}
L_{\mathrm{LR}}(\mathbf{w}; x, y) &= -\log\, P\left(y | x; \mathbf{w}\right) \\
&= -\log\, \frac{\exp(\mathbf{w}^\top f(x,y))}{\sum_{y' \in \mathcal{Y}(x)} \exp(\mathbf{w}^\top f(x,y'))} \\
&= -\mathbf{w}^\top f(x,y) + \log Z(\mathbf{w}, x)
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- Related loss functions: **hinge loss** (in SVM) and the **perceptron loss**.

Main Loss Functions: Summary

Squared (linear regression)	$\frac{1}{2}\left(y - \mathbf{w}^\top \mathbf{f}(x)\right)^2$
Log-linear (MaxEnt, CRF, logistic)	$-\mathbf{w}^\top \mathbf{f}(x,y) + \log \sum_{y' \in \mathcal{Y}} \exp(\mathbf{w}^\top \mathbf{f}(x,y'))$
Hinge (SVMs)	$-\mathbf{w}^\top \mathbf{f}(x,y) + \max_{y' \in \mathcal{Y}} \left(\mathbf{w}^\top \mathbf{f}(x,y') + c(y,y')\right)$
Perceptron	$-\mathbf{w}^\top \mathbf{f}(x,y) + \max_{y' \in \mathcal{Y}} \mathbf{w}^\top \mathbf{f}(x,y')$

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The log-linear, hinge, and perceptron losses are particular cases of general family (Martins et al., 2010).

Regularization

Regularized parameter estimate:

$$\widehat{\mathbf{w}} = \arg\min_{\mathbf{w}} \Omega(\mathbf{w}) + \underbrace{\sum_{n=1}^{N} L(\mathbf{w}; x_n, y_n)}_{\text{total loss}}$$

where $\Omega(\mathbf{w}) \geq 0$ and $\lim_{\|\mathbf{w}\| \to \infty} \Omega(\mathbf{w}) = \infty$ (**coercive** function).

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Equivalent, under mild conditions (namely convexity).

Regularization vs. Bayesian estimation

Regularized parameter estimate: $\widehat{\mathbf{w}} = \arg\min_{\mathbf{w}} \Omega(\mathbf{w}) + \sum_{n=1}^{N} L(\mathbf{w}; x_n, y_n)$

...interpretable as Bayesian **maximum a posteriori** (MAP) estimate:

$$\widehat{\mathbf{w}} = \arg\max_{\mathbf{w}} \underbrace{\exp\left(-\Omega(\mathbf{w})\right)}_{\text{prior } p(\mathbf{w})} \underbrace{\prod_{n=1}^{N} exp\left(-L(\mathbf{w}; x_n, y_n)\right)}_{\text{likelihood (i.i.d. data)}}.$$

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- ... and the squared error (SE) loss: $L_{\mathrm{SE}}(\mathbf{w}; x_n, y_n) = \frac{1}{2}\left(y - \mathbf{w}^\top \mathbf{f}(x)\right)^2 = -\log \mathcal{N}(y|\mathbf{w}^\top \mathbf{f}(x), 1)$

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- Notable cases:
  - the famous ℓ_1 norm: $\|\mathbf{w}\|_1 = \sum_i |w_i|$
  - the ℓ_∞ norm: $\|\mathbf{w}\|_\infty = \lim_{p\rightarrow\infty} \|\mathbf{w}\|_p = \max\{|w_i|, \; i = 1, ..., D\}$

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- The infamous ℓ_0 "norm": $\|\mathbf{w}\|_0 = \lim_{p\to 0} \|\mathbf{w}\|_p^p = |\{i : w_i \neq 0\}|$

  (not a norm).

Classical Regularizers: Ridge

Regularized parameter estimate: $\widehat{\mathbf{w}} = \arg\min_{\mathbf{w}} \Omega(\mathbf{w}) + \sum_{n=1}^{N} L(\mathbf{w}; x_n, y_n)$

Arguably, the most classical choice: squared ℓ_2 norm: $\Omega(\mathbf{w}) = \frac{\lambda}{2}\|\mathbf{w}\|_2^2$

- Corresponds to zero-mean Gaussian prior $p(\mathbf{w}) \propto \exp\left(-\lambda\|\mathbf{w}\|_2^2\right)$

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- **Ridge logistic regression**: Schaefer et al. (1984), Cessie and Houwelingen (1992); in ML: Chen and Rosenfeld (1999).

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- **Ridge regression** (SE loss): Hoerl (1962), Hoerl and Kennard (1970).
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- Used early in computer vision (Bertero et al., 1988), (Poggio et al., 1985).
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Classical Regularizers: Lasso

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- Geology/geophysics
  - Claerbout and Muir (1973)
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from (Goyal et al, 2010)

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- **Pros**: encourages sparsity: embedded feature selection.
- **Cons**: convex, but non-smooth: more challenging optimization.

The Lasso and Sparsity

Why does the Lasso yield sparsity?

The simplest (1D) case:

$$\widehat{w} = \arg\min_{w} \frac{1}{2}(w-y)^2 + \lambda|w| = \text{soft}(y,\lambda) = \begin{cases} y-\lambda & \Leftarrow \quad y > \lambda \\ 0 & \Leftarrow \quad |y| \leq \lambda \\ y+\lambda & \Leftarrow \quad y < -\lambda \end{cases}$$

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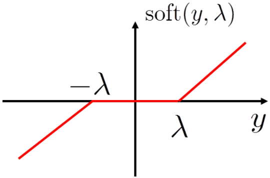

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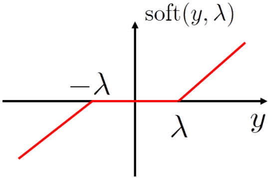

Contrast with the squared ℓ_2 (ridge) regularizer (linear scaling):

$$\widehat{w} = \arg\min_{w} \frac{1}{2}(w-y)^2 + \frac{\lambda}{2}w^2 = \frac{1}{1+\lambda}\,y$$

The Lasso and Sparsity (II)

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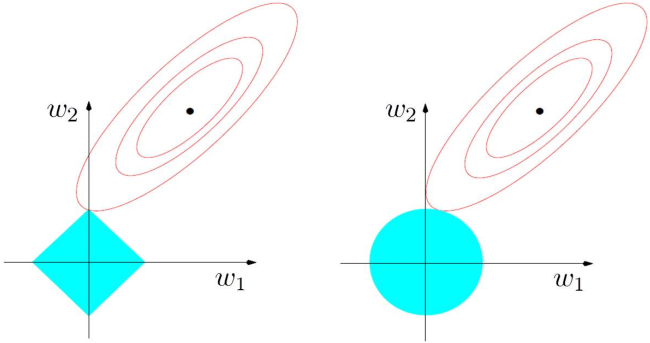

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Relevant Theory?

Theoretical results for ℓ_1-regularized logistic regression:

- PAC-Bayesian bounds (generalization improves with sparsity): Krishnapuram et al. (2005)
- Logarithmic sample complexity (versus linear, for ridge) Ng (2004)
- Oracle bounds (van de Geer, 2008)
- Consistency (Negahban et al., 2012)

Outline

Models

ℓ_1 regularization promotes **sparse models**

A very simple sparsity pattern: **small cardinality**

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A very simple sparsity pattern: **small cardinality**

Main question: how to promote less trivial sparsity patterns?

Structured Sparsity and Groups

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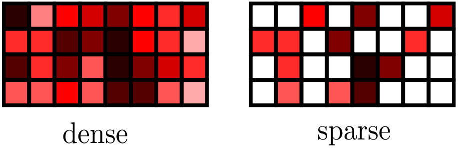
Many applications:

- feature template selection (Martins et al., 2011b)
- multi-task learning (Caruana, 1997; Obozinski et al., 2010)
- multiple kernel learning (Lanckriet et al., 2004)
- learning the structure of graphical models (Schmidt and Murphy, 2010)

"Grid" Sparsity

For feature spaces that can be arranged as a grid (examples next)



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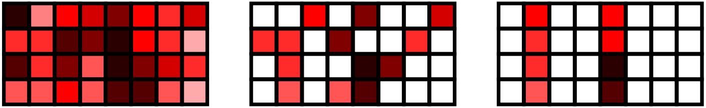

dense sparse **group sparse**

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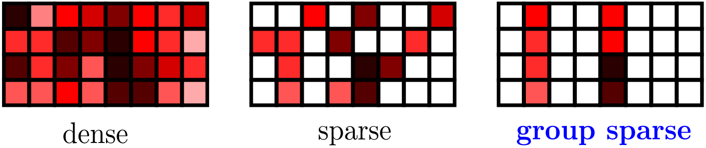


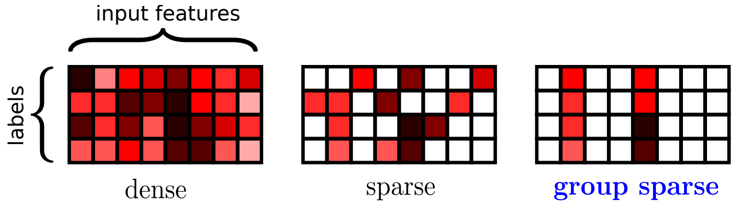
Goal: push *entire columns* to have zero weights

The groups are the columns of the grid

Example 1: Sparsity with Multiple Classes

Assume the feature map decomposes as $\mathbf{f}(x, y) = \mathbf{f}(x) \otimes \mathbf{e}_y$

In words: we're conjoining each input feature with each output class



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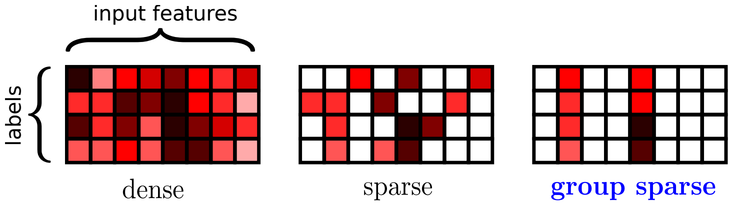


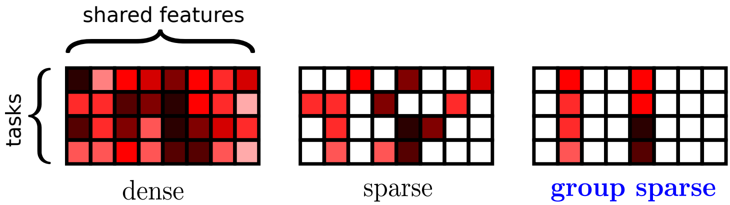
"Standard" sparsity is wasteful—we may still need all the input features

What we want: discard some input features

Solution: one group per *input* feature

Example 2: Multi-Task Learning (Caruana, 1997; Obozinski et al., 2010)

Same thing, except now rows are **tasks** and columns are **features**



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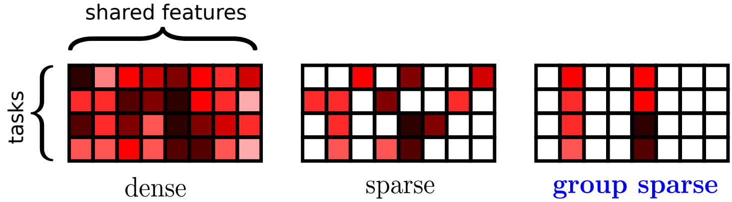

What we want: discard features that are irrelevant for *all* tasks

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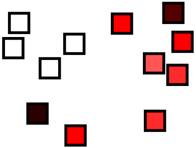
Group Sparsity



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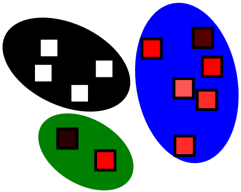

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- M groups $G_1, \ldots, G_M$, each $G_m \subseteq \{1, \ldots, D\}$
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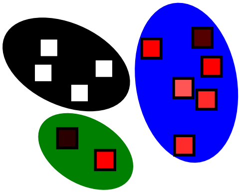

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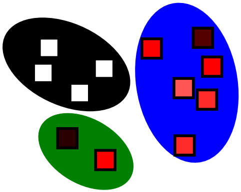

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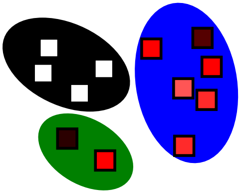

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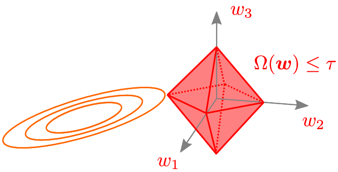
- λ_m: prior weight for group G_m (different groups have different sizes)

Lasso versus group-Lasso



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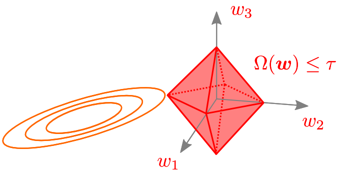

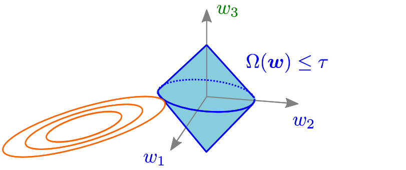

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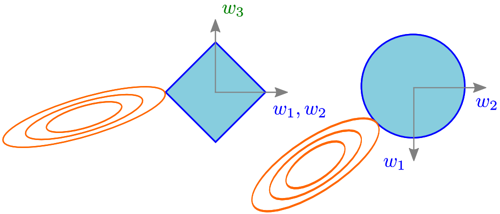

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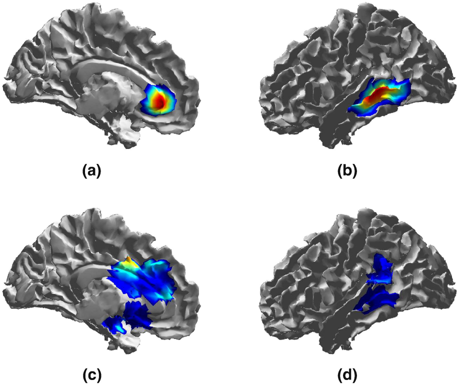
Examples of non-trivial groups:

- label-based groups (groups are columns of a matrix)
- template-based groups (next)

Example: Magnetoencephalographic (MEG) Reconstruction

Group: localized cortex area at localized time period (Bolstad et al., 2009)



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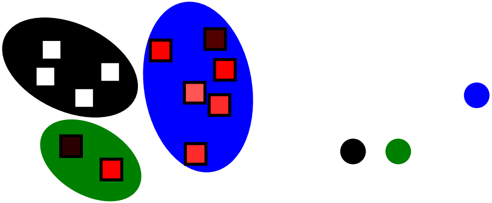

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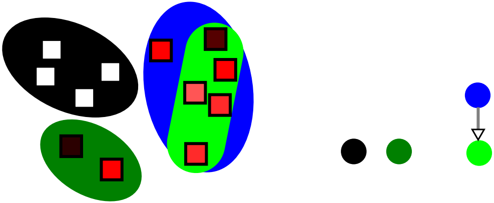

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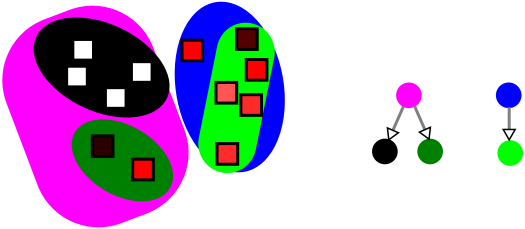

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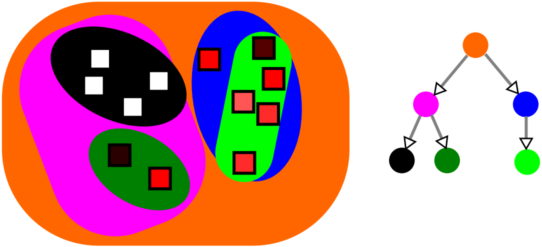

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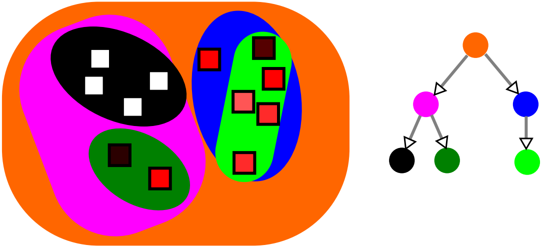

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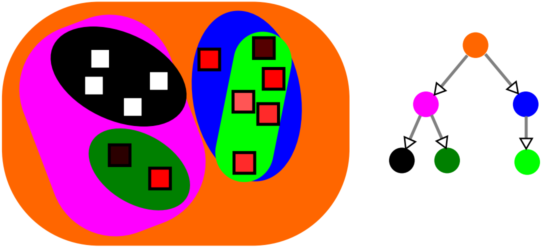

- What is the **sparsity pattern**?
- **If a group is discarded, all its descendants are also discarded**

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- **Choice of groups:** problem dependent, opportunity to use prior knowledge to favour certain structural patterns
- **Next:** algorithms
- We'll see that optimization is easier with non-overlapping or tree-structured groups than with arbitrary overlaps

Outline

Learning the Model

Recall that learning involves solving

$$\min_{\mathbf{w}} \quad \underbrace{\Omega(\mathbf{w})}_{\text{regularizer}} + \underbrace{\frac{1}{N}\sum_{i=1}^{N} L(\mathbf{w}, x_i, y_i)}_{\text{total loss}},$$

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Two kinds of optimization algorithms:

- batch algorithms (attacks the complete problem);
- online algorithms (use the training examples one by one)

Outline

A Key Ingredient: Proximity Operators

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$$\boxed{\mathbf{w} \mapsto \text{prox}_{\Omega}(\mathbf{w}) = \arg\min_{\mathbf{u}} \frac{1}{2}\|\mathbf{u} - \mathbf{w}\|_2^2 + \Omega(\mathbf{u})}$$

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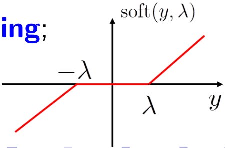
- ℓ_1 regularization, $\Omega(\mathbf{w}) = \lambda\|\mathbf{w}\|_1$: **soft-thresholding**;

$$\text{prox}_{\Omega}(\mathbf{w}) = \text{soft}(\mathbf{w}, \lambda)$$



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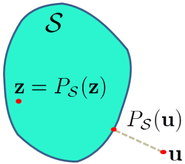
- indicator function, ${\color{red}\Omega(\mathbf{w}) = \iota_{\mathcal{S}}(\mathbf{w})} = \begin{cases} 0 & \Leftarrow \quad \mathbf{w} \in \mathcal{S} \\ +\infty & \Leftarrow \quad \mathbf{w} \notin \mathcal{S} \end{cases}$

$$\text{prox}_\Omega(\mathbf{w}) = P_{\mathcal{S}}(\mathbf{w})$$



Euclidean projection

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- **Arbitrary groups:**
  - For $\Omega_m(\mathbf{w}_m) = \|\mathbf{w}_m\|_2$: solved by convex smooth optimization (Yuan et al., 2011).
  - Sequential proximity steps (Martins et al., 2011a) (more later).

Outline

(Block-)Coordinate Descent

$\min_{\mathbf{w}} \ \Omega(\mathbf{w}) + \Lambda(\mathbf{w})$, where $\Lambda(\mathbf{w}) = \frac{1}{N}\sum_{i=1}^{N} L(\mathbf{w}, x_i, y_i)$ (loss)

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Update one (block of) component(s) of $\mathbf{w}$ at a time:

$$w_i^{\text{new}} \leftarrow \arg\min_{w_i} \Omega([w_1, ..., w_i, ...w_D]) + \Lambda([w_1, ..., w_i, ...w_D])$$

(Genkin et al., 2007; Krishnapuram et al., 2005; Liu et al., 2009; Shevade and Keerthi, 2003; Tseng and Yun, 2009; Yun and Toh, 2011)

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Shown to converge; competitive with state-of-the-art (Yun and Toh, 2011).

Has been used in NLP: Sokolovska et al. (2010); Lavergne et al. (2010).

Projected Gradient

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Shown later: **projected gradient** is a particular instance of the more general **proximal gradient** methods.

From Gradient to Hessian: Newton's Method

Assume $F(\mathbf{w}) = \Omega(\mathbf{w}) + \Lambda(\mathbf{w})$ is twice-differentiable.

Second order (quadratic) Taylor expansion around $\mathbf{w}'$:

$$F(\mathbf{w}) \approx F(\mathbf{w}') + \underbrace{\nabla F(\mathbf{w}')}_{\textbf{Gradient}}^\top (\mathbf{w} - \mathbf{w}') + \frac{1}{2}(\mathbf{w} - \mathbf{w}')^\top \underbrace{\mathbf{H}(\mathbf{w}')}_{\textbf{Hessian:}} (\mathbf{w} - \mathbf{w}')$$

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Quasi-Newton methods, namely **L-BFGS**, *approximate* the inverse Hessian directly from past gradient information.

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Can be derived with different tools:

- expectation-maximization (EM) (Figueiredo and Nowak, 2003);
- majorization-minimization (Daubechies et al., 2004);
- forward-backward splitting (Combettes and Wajs, 2006);
- separable approximation (Wright et al., 2009).

Monotonicity and Convergence

Proximal gradient, a.k.a., iterative shrinkage thresholding (IST):

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Important: monotonicity doesn't imply convergence of $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_t, \ldots$.

Convergence (even with inexact steps) proved for $\eta_t \leq 2/L$ (Combettes and Wajs, 2006).

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Resulting algorithm: **SpaRSA** (sparse reconstruction by separable approximation); shown to converge and to be fast Wright et al. (2009).

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$$\boxed{\begin{array}{lcl} b_{t+1} & = & \frac{1+\sqrt{1+4\,b_t^2}}{2} \\ \mathbf{z} & = & \mathbf{w}_t + \frac{b_t-1}{b_{t+1}}\left(\mathbf{w}_t - \mathbf{w}_{t-1}\right) \\ \mathbf{w}_{t+1} & = & \mathrm{prox}_{\eta\Omega}\left(\mathbf{z} - \eta\nabla\Lambda(\mathbf{z})\right) \end{array}}$$

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Convergence of iterates has not been shown.

Least Angle Regression (LARS)

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Key ideas (Efron et al., 2004; Osborne et al., 2000)

- "regularization path" $\hat{\mathbf{w}}(\lambda)$ is piecewise linear (Markowitz, 1952);
- the cusps can be identified in closed form;
- simply jump from one cusp to the next.

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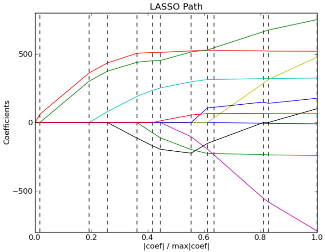
Key ideas (Efron et al., 2004; Osborne et al., 2000)

- "regularization path" $\hat{\mathbf{w}}(\lambda)$ is piecewise linear (Markowitz, 1952);
- the cusps can be identified in closed form;
- simply jump from one cusp to the next.



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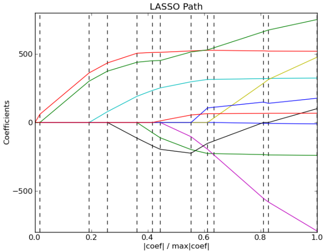

Cons: doesn't apply to group regularizers; exponential worst case complexity (Mairal and Yu, 2012).

Homotopy/Continuation Methods

LARS is related to a more general family: homotopy/continuation methods.

Consider $\boxed{\hat{\mathbf{w}}(\lambda) = \arg\min_{\mathbf{w}} \; \lambda\bar{\Omega}(\mathbf{w}) + \Lambda(\mathbf{w})}$

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- start with high value of λ, such that $\hat{\mathbf{w}}(\lambda)$ is easy (*e.g.*, zero);
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It's a meta-algorithm of general applicability when using "warm startable" solvers (Figueiredo et al., 2007; Hale et al., 2008; Osborne et al., 2000).

Some Stuff We Didn't Talk About

- shooting method (Fu, 1998)
- grafting (Perkins et al., 2003) and grafting-light (Zhu et al., 2010)
- orthant-wise limited-memory quasi-Newton (OWL-QN) (Andrew and Gao, 2007; Gao et al., 2007); doesn't handle overlapping groups
- alternating direction method of multipliers (ADMM); handles overlapping groups (Afonso et al., 2010; Figueiredo and Bioucas-Dias, 2011).
- forward stagewise regression (Hastie et al., 2007);

...several more; this is an active research area!

Outline

Why Online?

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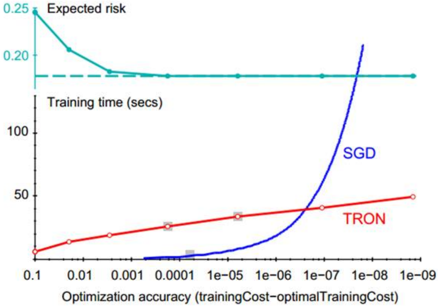
3. Faster to approach a near-optimal region
4. Slower convergence, but this is fine in machine learning ("the tradeoffs of large scale learning" by Bottou and Bousquet (2007))



Plain Stochastic (Sub-)Gradient Descent

$$\min_{\mathbf{w}} \quad \underbrace{\Omega(\mathbf{w})}_{\text{regularizer}} + \underbrace{\frac{1}{N}\sum_{i=1}^{N} L(\mathbf{w}, x_i, y_i)}_{\text{empirical loss}},$$

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input: stepsize sequence $(\eta_t)_{t=1}^{T}$
initialize $\mathbf{w} = \mathbf{0}$
for $t = 1, 2, \ldots$ **do**
  take training pair (x_t, y_t)
  (sub-)gradient step: $\mathbf{w} \leftarrow \mathbf{w} - \eta_t \left(\tilde{\nabla}\Omega(\mathbf{w}) + \tilde{\nabla} L(\mathbf{w}; x_t, y_t) \right)$
end for

SGD with Sparsity?

(Sub-)gradient step: $\boxed{\mathbf{w} \;\leftarrow\; \mathbf{w} - \eta_t \left(\tilde{\nabla}\Omega(\mathbf{w}) + \tilde{\nabla} L(\mathbf{w}; x_t, y_t)\right)}$

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- ℓ_2-regularization $\boxed{\Omega(\mathbf{w}) = \frac{\lambda}{2}\|\mathbf{w}\|_2^2} \Longrightarrow \tilde{\nabla}\Omega(\mathbf{w}) = \lambda \mathbf{w}$

$$\mathbf{w} \;\leftarrow\; \underbrace{(1 - \eta_t \lambda)\mathbf{w}}_{\text{scaling}} - \eta_t \tilde{\nabla} L(\mathbf{w}; x_t, y_t)$$

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$$\mathbf{w} \leftarrow \underbrace{\mathbf{w} - \eta_t \lambda \mathrm{sign}(\mathbf{w})}_{\text{constant penalty}} - \eta_t \tilde{\nabla} L(\mathbf{w}; x_t, y_t)$$

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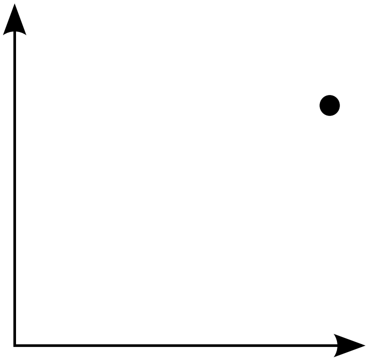
- **Problem: iterates are never sparse!**

Plain SGD with ℓ_2-regularization



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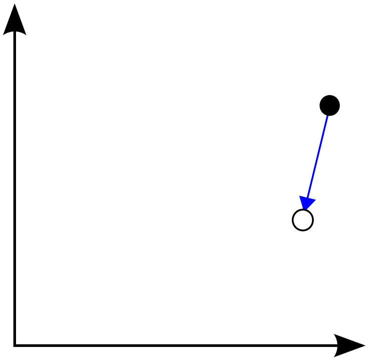

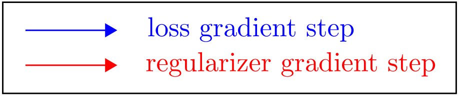

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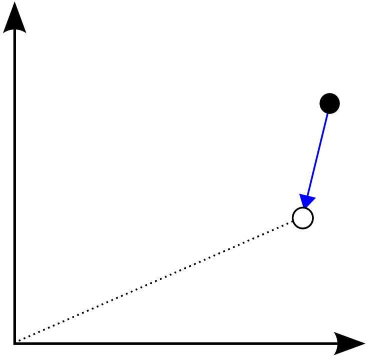

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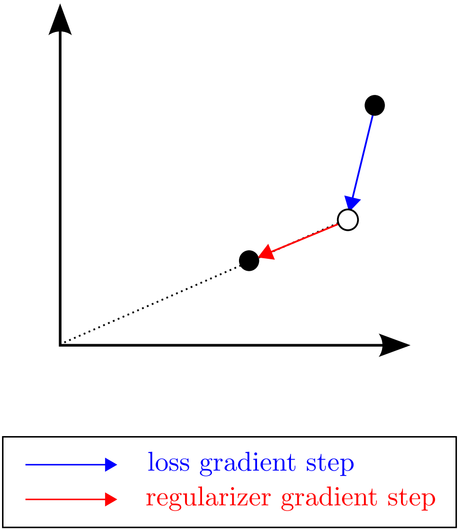

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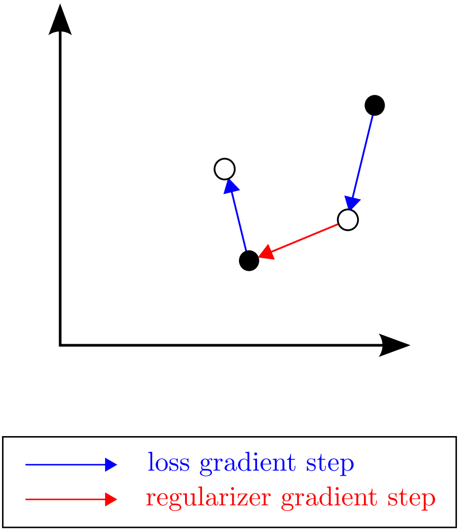

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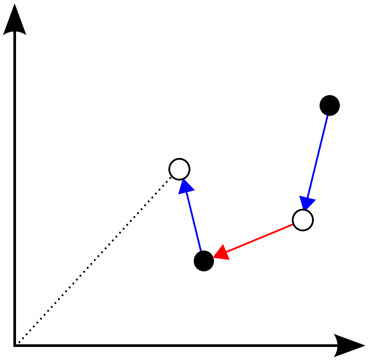

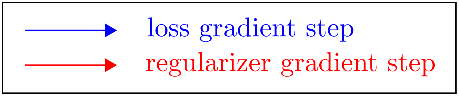

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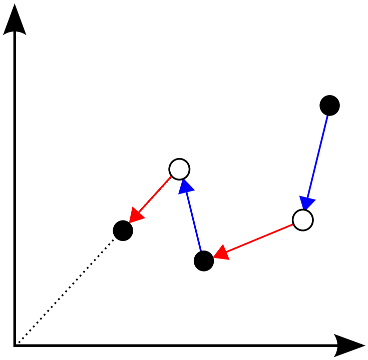

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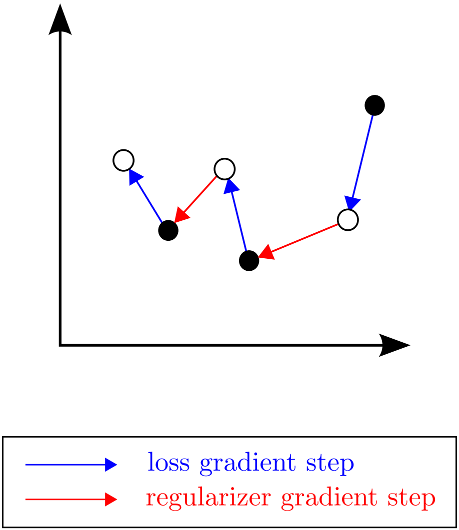

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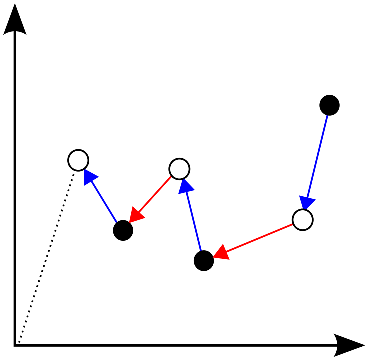

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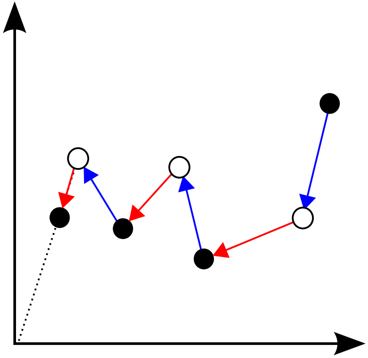

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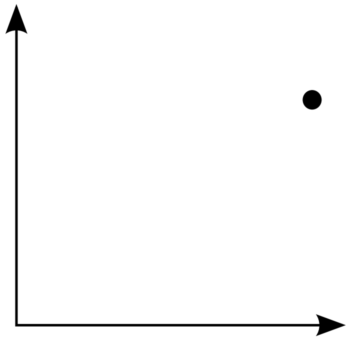

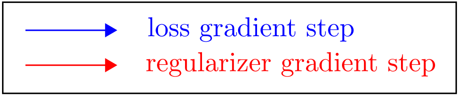

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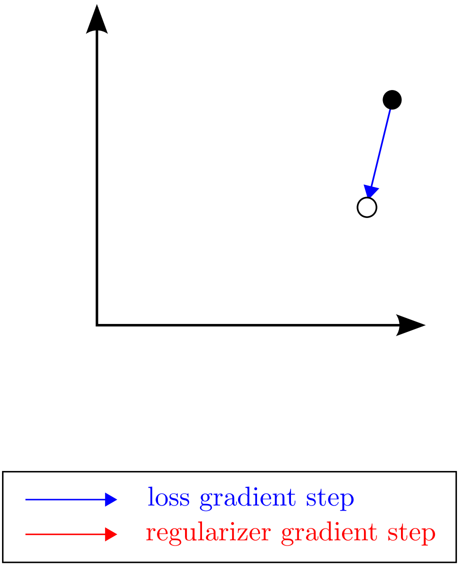

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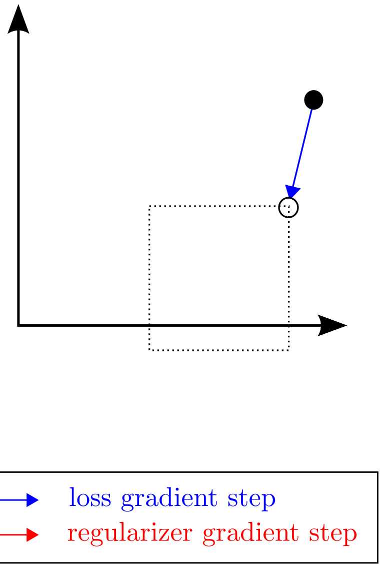

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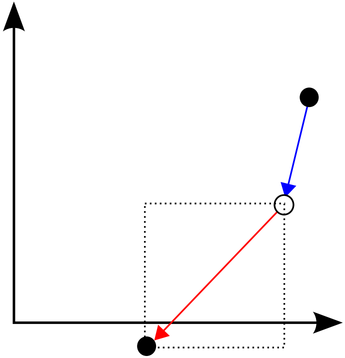

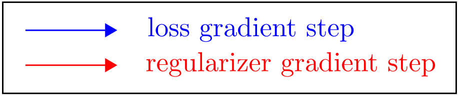

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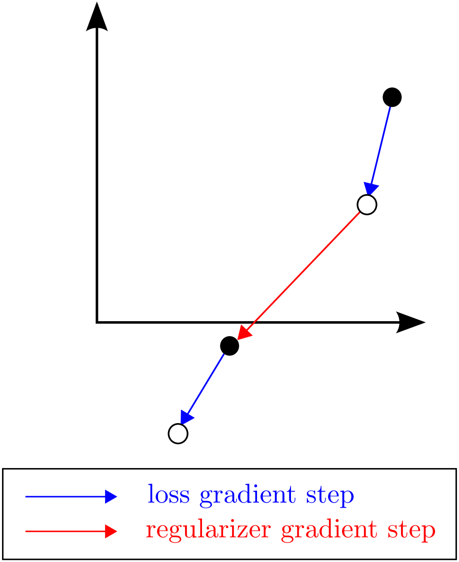

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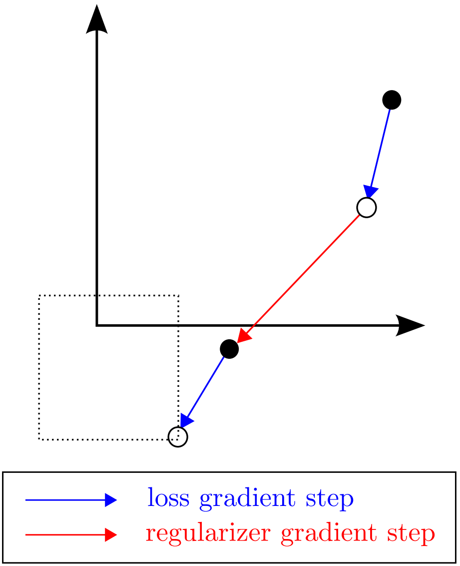

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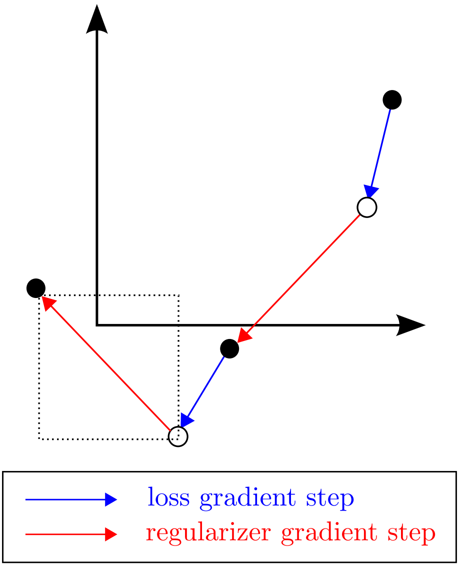

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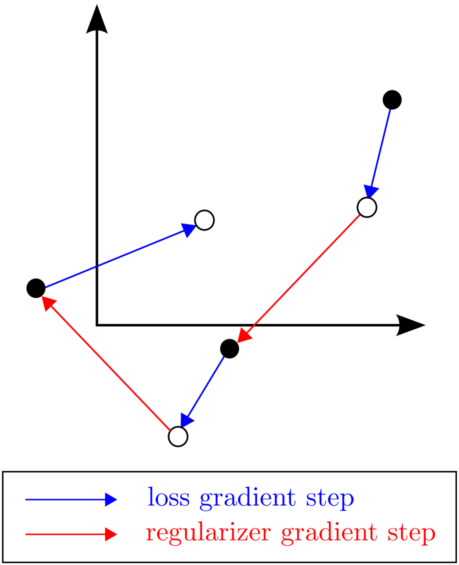

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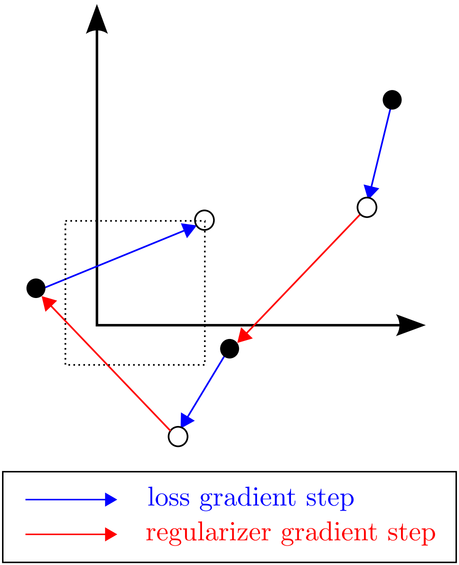

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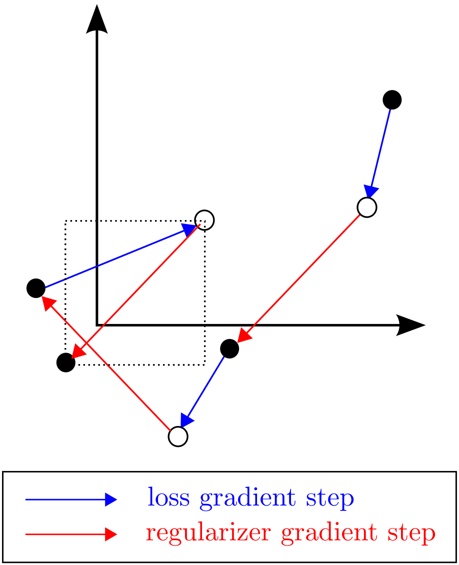

“Sparse” Online Algorithms

- Truncated Gradient (Langford et al., 2009)
- Online Forward-Backward Splitting (Duchi and Singer, 2009)
- Regularized Dual Averaging (Xiao, 2010)
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Truncated Gradient (Langford et al., 2009)

- take gradients-step **only with respect to the loss**
- apply soft-thresholding
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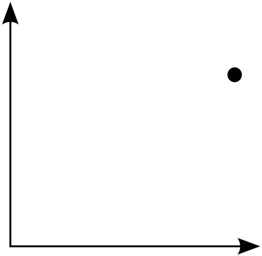

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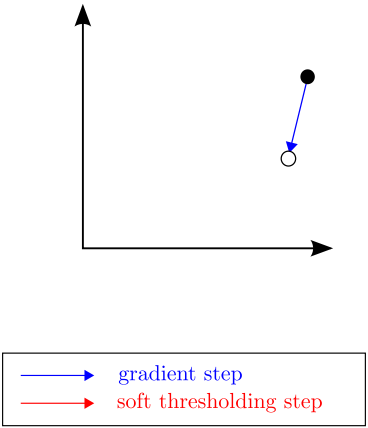

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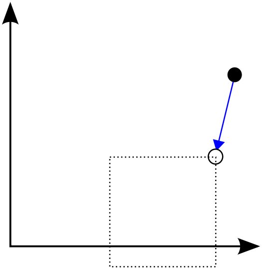

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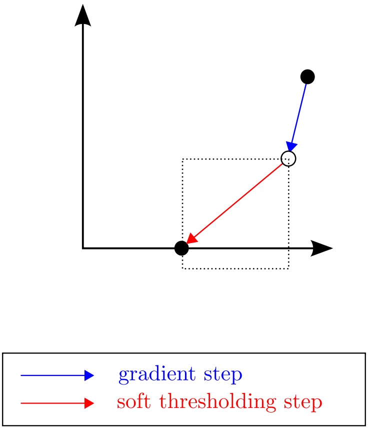

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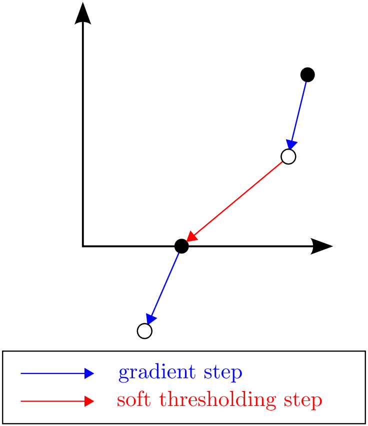

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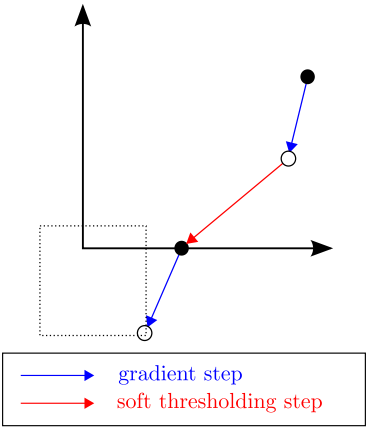

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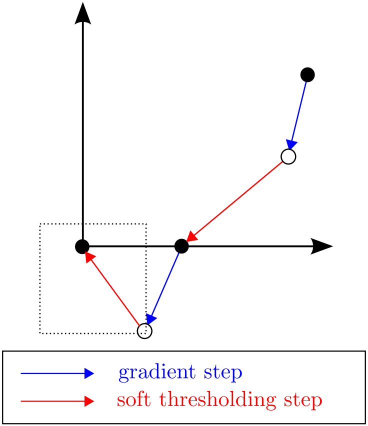

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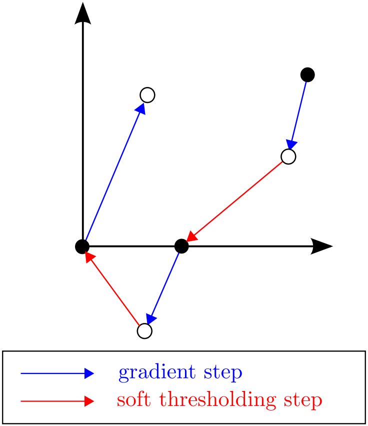

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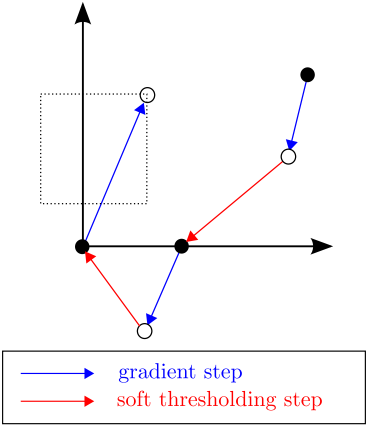

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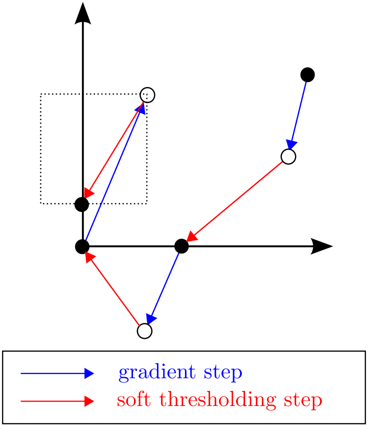

"Sparse" Online Algorithms

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Online Forward-Backward Splitting (Duchi and Singer, 2009)

input: stepsize sequences $(\eta_t)_{t=1}^T$, $(\rho_t)_{t=1}^T$
initialize $\mathbf{w} = \mathbf{0}$
for $t = 1, 2, \ldots$ **do**
  take training pair (x_t, y_t)
  gradient step: $\mathbf{w} \leftarrow \mathbf{w} - \eta_t \nabla L(\mathbf{w}; x_t, y_t)$
  proximal step: $\mathbf{w} \leftarrow \text{prox}_{\rho_t \Omega}(\mathbf{w})$
end for

- generalizes truncated gradient to arbitrary regularizers Ω
  - can tackle non-overlapping or hierarchical group-Lasso, but arbitrary overlaps are difficult to handle (more later)
- **practical drawback:** iterates are usually not very sparse
- **converges to ϵ-accurate objective after $O(1/\epsilon^2)$ iterations**

What About Group Sparsity?

Online forward-backward splitting (Duchi and Singer, 2009) handles groups.

All that is necessary is to compute $\text{prox}_{\boldsymbol{\Omega}}(\mathbf{w})$

- easy for non-overlapping and tree-structured groups
- **But what about general overlapping groups?**

Martins et al. (2011a): a prox-grad algorithm that can handle arbitrary overlapping groups

- decompose $\Omega(\mathbf{w}) = \sum_{j=1}^{J} \Omega_j(\mathbf{w})$ where each Ω_j is **non-overlapping**
- then apply prox_{Ω_j} *sequentially*
- still convergent (Martins et al., 2011a)

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Online Proximal Gradient (Martins et al., 2011a)

input: gravity sequence $(\sigma_t)_{t=1}^{T}$, stepsize sequence $(\eta_t)_{t=1}^{T}$
initialize $\mathbf{w} = \mathbf{0}$
for $t = 1, 2, \ldots$ **do**
  take training pair (x_t, y_t)
  gradient step: $\mathbf{w} \leftarrow \mathbf{w} - \eta_t \nabla L(\theta; x_t, y_t)$
  sequential proximal steps:
  for $j = 1, 2, \ldots$ **do**
    $\mathbf{w} \leftarrow \operatorname{prox}_{\eta_t \sigma_t \Omega_j}(\mathbf{w})$
  end for
end for

Online Proximal Gradient (Martins et al., 2011a)

input: gravity sequence $(\sigma_t)_{t=1}^{T}$, stepsize sequence $(\eta_t)_{t=1}^{T}$
initialize $\mathbf{w} = \mathbf{0}$
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  for $j = 1, 2, \ldots$ **do**
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  end for
end for

- **PAC Convergence:** ϵ-accurate solution after $T \leq O(1/\epsilon^2)$ rounds
- **Computational efficiency:** Each gradient step is **linear** in the number of features.
  Each proximal step is **linear** in the number of groups M.

Summary of Algorithms

	Converges	Rate	Sparse	Groups	Overlaps
Coord. desc.	✓	?	✓	Maybe	No
Prox-grad	✓	$O(1/\epsilon)$	Yes/No	✓	Not easy
OWL-QN	✓	?	Yes/No	No	No
SpaRSA	✓	$O(1/\epsilon)$ or better	Yes/No	✓	Not easy
FISTA	✓	$O(1/\sqrt{\epsilon})$	Yes/No	✓	Not easy
ADMM	✓	$O(1/\epsilon)$, $O(1/\sqrt{\epsilon})$	No	✓	✓
Online subgrad.	✓	$O(1/\epsilon^2)$	No	✓	No
Truncated grad.	✓	$O(1/\epsilon^2)$	✓	No	No
FOBOS	✓	$O(1/\epsilon^2)$	Sort of	✓	Not easy
Online prox-grad	✓	$O(1/\epsilon^2)$	✓	✓	✓

Outline

Applications of Structured Sparsity in ML

Relatively few to date; we will focus on several recent NLP applications (Martins et al., 2011b):

- Phrase chunking
- Named entity recognition
- Dependency parsing

Martins et al. (2011b): Group by Template

Feature templates provide a straightforward way to define non-overlapping groups.

To achieve group sparsity, we optimize:

$$\min_{\mathbf{w}} \underbrace{\frac{1}{N}\sum_{n=1}^{N} L(\mathbf{w}; x_n, y_n)}_{\text{empirical loss}} + \underbrace{\Omega(\mathbf{w})}_{\text{regularizer}}$$

where we use the $\ell_{2,1}$ norm (group Lasso):

$$\Omega(\mathbf{w}) = \lambda \sum_{m=1}^{M} d_m \|\mathbf{w}_m\|_2$$

for M groups/templates.

Chunking

$x =$ "He reckons the current account deficit will narrow to only \$ 1.8 billion in September"

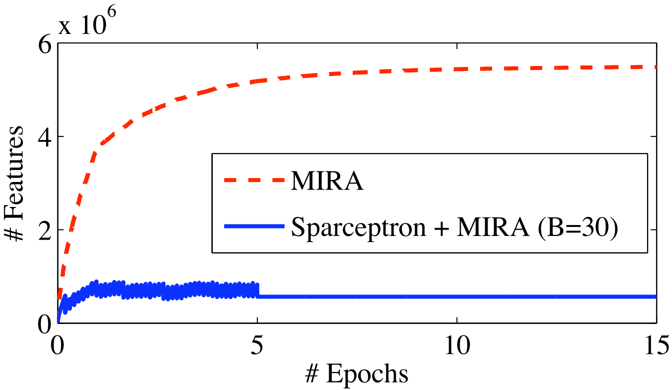
[NP He] [VP reckons] [NP the current account deficit] [VP will narrow] [PP to] [NP only # 1.8 billion] [PP in] [NP September]

- CoNLL 2000 shared task (Sang and Buchholz, 2000)
- Unigram features: 96 feature templates using POS tags, words, and word shapes, with various context sizes
- Bigram features: 1 template indicating the label bigram
- **Baseline**: L_2-regularized MIRA, 15 epochs, all features, cross-validation to choose regularization strength
- **Template-based group lasso**: 5 epochs of sparseptron + 10 of MIRA

Chunking Experiments

	Baseline	Template-based group lasso			
# templates	96	10	20	30	40
model size	5,300,396	71,075	158,844	389,065	662,018
F_1 (%)	93.10	92.99	93.28	**93.59**	93.42

Chunking



Memory requirement of sparseptron is $<$ 7.5% of that of the baseline.
(Oscillations are due to proximal steps after every 1,000 instances.)

Applications of Structured Sparsity in ML

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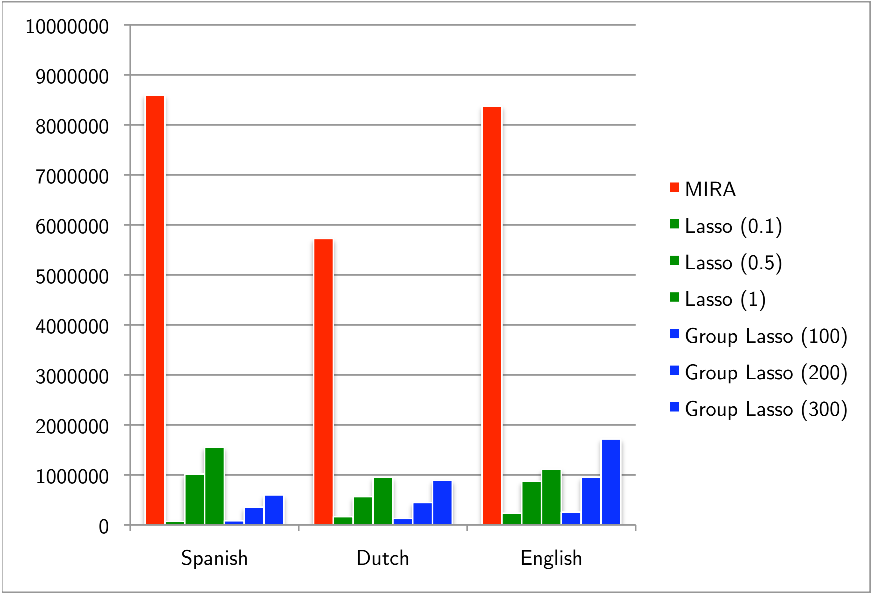
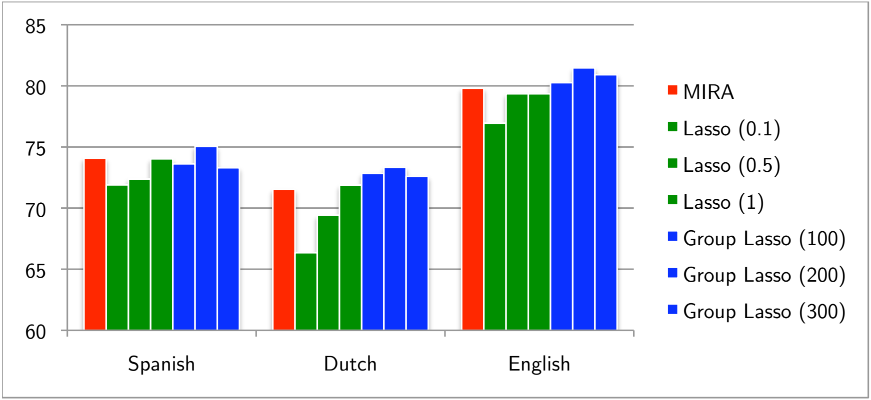
Named Entity Recognition

x	Only	France	and	Britain	backed	Fischler	's	proposal
	RB	NNP	CC	NNP	VBD	NNP	POS	NN
y = h_w(x)	O	I-LOC	O	I-LOC	O	I-PER	O	O

- CoNLL 2002/2003 shared tasks (Sang, 2002; Sang and De Meulder, 2003): Spanish, Dutch, and English
- Unigram features: 452 feature templates using POS tags, words, word shapes, prefixes, suffixes, and other string features, all with various context sizes
- Bigram features: 1 template indicating the label bigram
- **Baselines**:
  - L_2-regularized MIRA, 15 epochs, all features, cross-validation to choose regularization strength
  - sparseptron with lasso, different values of C
- **Template-based group lasso**: 5 epochs of sparseptron + 10 of MIRA



Named entity models: number of features. (Lasso $C = 1/\lambda N$.)



Named entity models: F_1 accuracy on the test set. (Lasso $C = 1/\lambda N$.)

Applications of Structured Sparsity in ML

Relatively few to date; we will focus on several recent NLP applications (Martins et al., 2011b):

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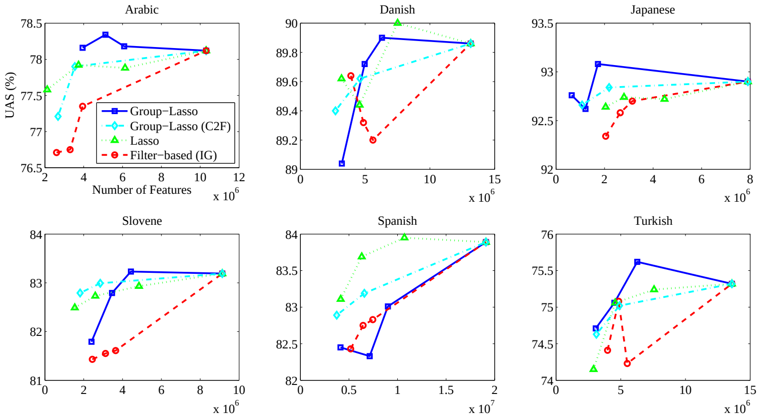
Non-projective Dependency Parsing

- CoNLL-X shared task (Buchholz and Marsi, 2006): Arabic, Danish, Dutch, Japanese, Slovene, and Spanish
- Arc-factored models (McDonald et al., 2005)
- 684 feature templates by conjoining words, shapes, lemmas, and POS of the head and the modifier, contextual POS, distance and attachment direction
- **Baselines**:
  - MIRA with all features
  - filter-based template selection (information gain)
  - standard lasso
- **Our methods**: template-based group lasso; coarse-to-fine regularization
- Budget sizes: 200, 300, and 400

Non-projective Dependency Parsing (c'ed)



Template-based group lasso is better at selecting feature templates than the IG criterion, and slightly better than **coarse-to-fine**.

Outline

Summary

- Sparsity is desirable in machine learning: *feature selection*, *runtime*, *memory footprint*, *interpretability*
- Beyond plain sparsity: **structured sparsity** can be promoted through group-Lasso regularization
- Choice of groups reflects prior knowledge about the desired sparsity patterns.
- Small/medium scale: many batch algorithms available, with fast convergence (IST, FISTA, SpaRSA, ...)
- Large scale: online proximal-gradient algorithms suitable to explore large feature spaces

Thank you!

- Questions?

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